LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2023**

PMT 2502 – MEASURE THEORY AND INTEGRATION

Date: 02-05-2023 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

Answer ALL Questions:

1. a) Show that the outer measure is a translation invariant. (5 marks)

OR

b) If f is a real-valued measurable function, then prove that the set $\{x: f(x) = \alpha\}$ is measurable for each extended real number α. (5 marks)

c) If $\mathcal{M}$ denotes the class of Lebesgue measurable sets, then verify whether $\mathcal{M}$ is a σ-algebra. (15 marks)

OR

d) Prove that there exists a non-measurable set. (15 marks)

2. a) If f is a measurable function, g is an integrable function and α, β are real numbers such that $\alpha \leq f \leq \beta$ a.e., then prove that there exists γ, $\alpha \leq \gamma \leq \beta$ such that $\int f[g]dx = \gamma \int [g]dx$. (5 marks)

OR

b) State and prove Lebesgue Monotone Convergence theorem. (5 marks)

c) Evaluate $\int_0^\infty \frac{sint}{e^t - x} dt, x \in [-1, 1]$. (15 marks)

OR

d) Prove that Riemann integrability implies Lebesgue integrability. (15 marks)

3. a) Prove that every σ-algebra is a σ-ring, but the converse is not true. (5 marks)

OR

b) State and prove Tchebychev's inequality. (5 marks)

c) If μ is a measure on a ring $\mathcal{R}$ and the set function μ^* defined on $\mathcal{H}(\mathcal{R})$ is given by $\mu^*(\text{E}) = \inf\{\sum_{i=1}^{\infty} \mu(\text{E}_i) : \text{E}_i \in \mathcal{R}, E \subseteq \bigcup_{i=1}^{\infty} \text{E}_i\}$, then prove that $\mu^*(E) = \mu(E)$ for $E \in \mathcal{R}$ and μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$. (15 marks)

OR

d) Prove that $\bar{\mathcal{S}}$ is a $\sigma - ring$ where μ is a measure defined on a $\sigma - ring\ \mathcal{S}$ and $\bar{\mathcal{S}} = \{E \Delta N : \in \mathcal{S}$ and N is contained in some set in $\mathcal{S}$ $with$ zero measure$\}$. Also, prove that the set function $\bar{\mu}$ defined by $\bar{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$. (15 marks)

4. a) Prove that a function ψ is convex if and only if ψ is continuous and midpoint convex. (5 marks)

OR

b) State and prove Cauchy Schwarz inequality. (5 marks)

c) State and prove Minkowski's inequality. State the condition for equality and prove it. (15 marks)

OR

d) (i) State and prove Completeness theorem for convergence in measure. (9 marks)

(ii) Let $\{f_n\}$ be a sequence of non negative measurable functions such that $|f_n| < g$, is an integrable function and $f_n \to f$ in measure. Then prove that f is integrable, $\lim \int f_n d\mu = \int f d\mu$ and $lim \int |f_n - f| \, d\mu = 0$. (6 marks)

5. a) For a signed measure ν defined on a measurable space $[\![X, S]\!]$, prove that there exists a positive set A and a negative set B such that $A \cup B = X$ and $A \cap B = \emptyset$ (5 marks)

OR

b) Let μ, λ, ν be σ-finite signed measures on $[\![X, S]\!]$ such that $\nu \ll \mu$, $\mu \ll \lambda$.Then show that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} [\lambda]$. (5 marks)

c) State and prove Jordan decomposition theorem. (15 marks)

OR

d) If $[\![X, S, \mu]\!]$ is a σ- finite measure space and γ is a σ- finite measure on S such that $\nu \ll \mu$, then prove there exists a finite valued non negative measurable function f on X such that for each $E \in S$, $\gamma(E) = \int_E f d\mu$. Also prove that f is unique in the sense that if $\gamma(E) = \int_E g d\mu$ for each $E \in S$, then $f = g$ a. e. (μ). (15 marks)

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